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Partition Complete Boolean Algebras and Almost Compact Cardinals

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Abstract. For an infinite cardinal κ a stronger version of κ-distributivity for Boolean algebras, called κ-partition completeness, is defined and investigated (e. g. every κ-Suslin algebra is a κ-partition complete Boolean algebra). It is shown that every κ-partition complete Boolean algebra is κ-weakly representable, and for strongly inaccessible κ these concepts coincide. For regular $\kappa \geq \omega$, it is proved that an atomless κ-partition complete Boolean algebra is an updirected union of basic κ-tree algebras. Using κ-partition completeness, the concept of γ-almost compactness is introduced for $\gamma \geq \kappa$. For strongly inaccessible κ we show that κ is κ-almost compact iff κ is weakly compact, and if κ is 2^κ-almost compact, then κ is measurable. Further κ is strongly compact iff it is γ-almost compact for all $\gamma \geq \kappa$.

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Introduction

Several definitions and results in set theory are motivated by a desire to generalise the following famous results about ω to larger cardinals:

K_ω: König's infinity lemma: Every tree of height ω in which every level has $< \omega$ elements contains a branch with ω elements.

S_ω: Stone's representation theorem: Every (ω-complete) Boolean algebra is isomorphic to a (ω-complete) subalgebra of some powerset algebra.

G_ω: Gödel's compactness theorem: Every ω-satisfiable subset of $L_{\omega\omega}$ is satisfiable.

R_ω: Ramsey's theorem: For any set X with $|X| \geq \omega$, if the set $[X]^n$ of all unordered n-tuples of X is partitioned into finitely many blocks, then there is a set $Y \subseteq X$ with $|Y| \geq \omega$ such that $[Y]^n$ is a subset of one of the blocks.

U_ω: The ultrafilter theorem: Every (ω-complete) meet semilattice of nonzero elements of a Boolean algebra can be extended to a (ω-complete) ultrafilter.

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In the simplest cases one replaces ω with an arbitrary cardinal κ and asks for which κ the properties K_κ, S_κ, G_κ, R_κ or U_κ hold. The aim is to characterise these cardinals in different ways or to prove some implications between these properties or variations of them. The idea of defining large cardinals via the existence of certain ultrafilters in Boolean algebras probably has its roots in [3] and [6]. In this paper we give such "ultrafilter definitions" for cardinals with the tree property, and for what we call almost compact cardinals (see Theorem 3.4 and Corollary 3.7).

The partition lattice of a set X has been studied extensively in set theory, lattice theory and combinatorics. Partitions of X are precisely the maximal antichains in the powerset algebra $\mathcal{P}(X)$, and this is one way of defining (abstract) partitions for arbitrary Boolean algebras. Although there are some results concerning partitions in some special Boolean algebras (see e. g. [10, Chapter 8, 9]), the general theory of partitions in Boolean algebras has received comparatively little attention to date. We hope that our concept of κ-partition completeness will convince the reader that it is worthwhile to study properties of partitions in arbitrary Boolean algebras.

Since the material of this paper is both algebraic and set theoretic, we will assume familiarity with only the basics of Boolean algebra and large cardinals (e. g. some knowledge of homomorphisms, filters, atomless Boolean algebras as well as regular and inaccessible cardinals should suffice). Standard references are [5] and [7].

After some preliminary results about partitions in Boolean algebras, the paper consists of three main sections that focus on the relation between partition completeness and weak representability, on atomless partition complete Boolean algebras, and on almost compactness respectively. The final section contains some open problems about almost compactness which we hope will stimulate further research in this direction.

0 Preliminaries: Partitions in Boolean algebras

0.1 Partition semilattices of Boolean algebras

Let B^+ be the set of nonzero elements of a Boolean algebra B. An *antichain* in B is a subset of pairwise disjoint elements of B^+. A subset σ of B^+ is said to be a *partition* of (or in) B if σ is an antichain and the join of σ exists and is the top element of B, i.e., $\Sigma\sigma = 1$ and $a \cdot b = 0$ for any distinct $a, b \in \sigma$. Note that σ is a partition if and only if it is a maximal antichain. A partition σ is said to be *finer* than (or a *refinement* of) a partition τ (in symbols $\sigma \leq \tau$) if for any $x \in \sigma$ there is a $y \in \tau$ such that $x \leq y$. We let $\mathbb{P}_B$ be the family of all partitions of B partially ordered by the refinement relation between partitions. The *coarsest refinement* or *meet* of a set of partitions in B is the greatest lower bound in this poset (if it exists). We use the symbol $\otimes$ for the meet of partitions. Thus for a set S of partitions $\bigotimes S$ is the meet (if it exists). Since Boolean algebras are distributive, the meet of any two partitions $\sigma, \tau \in \mathbb{P}_B$ is given by

$$\sigma \otimes \tau = \{x \cdot y : x \in \sigma,\ y \in \tau \text{ and } x \cdot y \neq 0\},$$

hence $\mathbb{P}_B$ is always a meet-semilattice. However, it is well-known that if B is a powerset algebra $\mathcal{P}(X)$ for some set X, then $\mathbb{P}_B$ forms a complete lattice (in fact an algebraic lattice, called the *partition lattice on* X). The following definitions and results attempt to show that, by imposing some completeness criteria on arbitrary Boolean algebras, the observation about the powerset algebras can be partially extended to the more general setting.

We assume throughout this paper that κ *is an infinite cardinal.* For a Boolean algebra B we let

$$\mathbb{P}_B^\kappa = \{\sigma \in \mathbb{P}_B : |\sigma| < \kappa\}.$$

Note that for $\sigma, \tau \in \mathbb{P}_B$ we have $|\sigma| \leq |\tau|$ whenever τ refines σ. Therefore $\mathbb{P}_B^\kappa$ is an up-closed subset of $\mathbb{P}_B$ and hence, for any subset S of $\mathbb{P}_B^\kappa$, the join of S in $\mathbb{P}_B^\kappa$ coincides with the join of S in $\mathbb{P}_B$ (if it exists). However, even finite joins may not exist in $\mathbb{P}_B$ (take for example the partitions $\sigma = \{\{0,2\},\{1,3\},\{4,6\},\{5,7\},\{8,10\},\ldots\}$ and $\tau = \{\{0\},\{1,3\},\{2,4\},\{5,7\},\{6,8\},\ldots\}$ in the Boolean algebra of finite and cofinite subsets of ω). The situation is somewhat better for meets: since

$$|\sigma \otimes \tau| \leq |\sigma \times \tau| = |\sigma| \cdot |\tau|,$$

$\mathbb{P}_B^\kappa$ is at least closed under finite meets.

Recall that a lattice is κ*-complete* if the join and the meet of any subset with less than κ elements exist. A sublattice L' of a lattice L is a κ*-complete sublattice* if L' is κ-complete and the joins and the meets of subsets with less than κ elements in L' and L coincide. If the lattice or sublattice has at most κ elements, we drop the reference to κ. Of course the same terminology applies to Boolean algebras, and in this case closure under κ-meets is equivalent to closure under κ-joins since complementation is an isomorphism from a Boolean algebra to its dual. A κ-complete subalgebra of a powerset algebra is refered to as a κ*-set algebra*.

Suppose now that B is κ-complete and $\sigma \in \mathbb{P}_B^\kappa$. Then all joins of subsets of σ exist, hence the powerset algebra $\mathcal{P}(\sigma)$ is isomorphic to a complete subalgebra of B. It is convenient to identify $\mathcal{P}(\sigma)$ with this complete subalgebra, so that we may drop the phrase "isomorphic to". This convention also allows us to observe that for any infinite cardinal $\lambda \leq \kappa$, any κ-complete Boolean algebra is an updirected union of powerset algebras: $B = \bigcup\{\mathcal{P}(\sigma) : \sigma \in \mathbb{P}_B^\lambda\}$.

Lemma 0.1. *Let B be a κ-complete Boolean algebra.*

(i) *For any $\sigma \in \mathbb{P}_B^\kappa$, the partition lattice $\mathbb{P}_{\mathcal{P}(\sigma)}$ is (isomorphic to) the principal filter $[\sigma)$ of $\mathbb{P}_B^\kappa$.*

(ii) *For any $S \subseteq \mathbb{P}_B^\kappa$, if $\bigotimes S$ exists in $\mathbb{P}_B^\kappa$, then the join of S exists in $\mathbb{P}_B^\kappa$.*

Proof. For (i) we observe that if $\tau \geq \sigma$, then each element of τ is a join of elements of σ. To prove (ii), let $\sigma = \bigotimes S \in \mathbb{P}_B^\kappa$. Then $S \subset \mathbb{P}_{\mathcal{P}(\sigma)}$, a complete lattice, so the join of S exists in $\mathbb{P}_{\mathcal{P}(\sigma)}$. By (i) this join coincides with the join in $\mathbb{P}_B^\kappa$. □

Corollary 0.2. *For any κ-complete Boolean algebra B, the set $\mathbb{P}_B^\kappa$, ordered by refinement, is a lattice.*

Proof. As observed earlier, $\mathbb{P}_B^\kappa$ is closed under finite meets, so by (ii) of the preceding lemma, it is also closed under finite joins. □

Note that $\mathbb{P}_B^\kappa$ has a bottom element iff B is atomic and has less than κ atoms. (Of course $\mathbb{P}_B^\kappa$ always has $\{1\}$ as top element.)

0.2 Completeness and partition completeness

A Boolean algebra B is said to be *κ-distributive* if it is κ-complete and for any cardinal $\lambda < \kappa$ and any set $\{a_{i,j} \in B : i, j < \lambda\}$ we have

$$\Pi_{i<\lambda}\Sigma_{j<\lambda}a_{i,j} = \Sigma_{f\in\lambda^\lambda}\Pi_{i<\lambda}a_{i,f(i)}.$$

It can be shown (see e. g. [7, p. 217]) that κ-distributivity is equivalent to the following condition: *Any collection of less than κ partitions from $\mathbb{P}^\kappa_B$ has a coarsest refinement (not necessarily in $\mathbb{P}^\kappa_B$)*. A Boolean algebra is *completely distributive* if it is κ-distributive for all κ. The following strengthening of κ-distributivity is central to our results.

Definition 0.3. We say that a Boolean algebra B is *κ-partition complete* if $\mathbb{P}^\kappa_B$ is a κ-complete meet semilattice, i. e., the coarsest refinement of any collection of less than κ partitions from $\mathbb{P}^\kappa_B$ exists *and* is in $\mathbb{P}^\kappa_B$.

Example 0.4.

(i) If $|X| < \kappa$, then the powerset algebra $\mathcal{P}(X)$ is κ-partition complete since $\mathcal{P}(X)$ is completely distributive and $\mathbb{P}^\kappa_{\mathcal{P}(X)} = \mathbb{P}_{\mathcal{P}(X)}$ in this case.

(ii) It is well-known that every κ-set algebra is κ-distributive (see [7, p. 216]). In Section 1 we show that for any strongly inaccessible κ, the concepts of κ-distributivity and κ-partition completeness coincide. Hence for strongly inaccessible κ, every κ-set algebra is κ-partition complete.

(iii) A complete atomless Boolean algebra B is called a *κ-Suslin algebra* if it is κ-distributive and every antichain has less than κ elements. Note that being a κ-Suslin algebra is equivalent to $\mathbb{P}^\kappa_B = \mathbb{P}_B$, so B is κ-partition complete.

Other examples of κ-partition complete Boolean algebras are the basic κ-tree algebras in Section 2.

The following example demonstrates that κ-partition completeness is strictly stronger than κ-distributivity. Let B be the (completely distributive) powerset algebra of the half-open unit interval $[0,1)$ and let σ_n be the partition of $[0,1)$ into n equal half-open intervals of length $1/n$. Then $S = \{\sigma_n : n \in \omega\}$ is a countable subset of $\mathbb{P}^{\aleph_1}_B$ (actually of $\mathbb{P}^\omega_B$) but $\bigotimes S = \{\{a\} : 0 \le a < 1\}$ is not in $\mathbb{P}^{\aleph_1}_B$. Hence B is not $\aleph_1$-partition complete.

Lemma 0.5. *If B is a κ-partition complete Boolean algebra, then B is κ-complete.*

Proof. Assume B is κ-partition complete, and consider a subset S of B with $|S| < \kappa$. Let $\sigma = \bigotimes\{\{b,-b\} : b \in S\}$. Then S is a subset of the complete Boolean algebra $\mathcal{P}(\sigma)$, so $c = \Pi\, S$ exists in $\mathcal{P}(\sigma)$. We claim that either $c = 0$ or $c \in \sigma$, whence $c \in B$ and therefore c is also the meet of S in B.

Suppose $c \neq 0$ and $c \notin \sigma$. Then there exists distinct $x, y \in \sigma$ such that $x, y \le c$. But now the partition $\sigma' = \{x + y\} \cup (\sigma \setminus \{x, y\})$ is a refinement of $\{b, -b\}$ for all $b \in S$, and $\sigma < \sigma' \in \mathbb{P}^\kappa_B$. Since σ is the coarsest refinement of the collection $\{\{b,-b\} : b \in S\}$, this is a contradiction. □

The above lemma is a special case of the more general result that any κ-bounded Boolean power of a κ-complete lattice with respect to a κ-partition complete Boolean algebra is κ-complete. It is implicitely in [11].

Every Boolean algebra B is ω-complete and $\mathbb{P}_B^\omega$ forms a lattice. By Corollary 0.2, if B is κ-complete, then $\mathbb{P}_B^\kappa$ is a lattice. The following result is a stronger version of this corollary.

Proposition 0.6. *A Boolean algebra B is κ-partition complete if and only if the poset $\mathbb{P}_B^\kappa$ is a κ-complete lattice.*

Proof. Suppose B is κ-partition complete and let S be a subset of $\mathbb{P}_B^\kappa$ with $|S| < \kappa$. Then $\bigotimes S$ exists in $\mathbb{P}_B^\kappa$, hence by Lemmas 0.1(ii) and 0.5, the join of S also exists. The converse follows immediately from the definition of κ-partition completeness. □

Recall that a Boolean algebra homomorphism is *κ-complete* if it preserves joins and meets of subsets with less than κ elements. A filter in a Boolean algebra is *κ-complete* if the meet of any subset with less than κ elements exists and is again in the filter. For later use we collect below some standard results about κ-completeness. The proofs are simple extensions of the ω-complete case (see e. g. [7]).

Lemma 0.7. *Let A, A_i $(i \in I)$ be κ-complete Boolean algebras, and suppose f is a surjective homomorphism from A onto a Boolean algebra B.*

(i) *f is κ-complete iff $\ker f$ is a κ-complete ideal.*

(ii) *If f is κ-complete, then B is κ-complete.*

(iii) *The direct product $\Pi_I A_i$ is κ-complete.*

(iv) *If F is a κ-complete filter on I, then the canonical surjective homomorphism from $\Pi_I A_i$ to the reduced product $\Pi_F A_i$ is κ-complete.*

(v) *If f is κ-complete and C is a κ-complete subalgebra of B, then the preimage $f^{-1}[C]$ is a κ-complete subalgebra of A.*

(vi) *Any κ-complete filter in a κ-complete subalgebra of A extends to a κ-complete filter in A.*

The next proposition shows that κ-partition completeness is preserved by κ-complete homomorphisms.

Proposition 0.8. *Let A and B be κ-complete Boolean algebras, and suppose $f : A \twoheadrightarrow B$ is a κ-complete surjective homomorphism.*

(i) *The map $\hat{f} : \mathbb{P}_A^\kappa \longrightarrow \mathbb{P}_B^\kappa$, defined by*

$$\hat{f}(\sigma) = \{f(x) : x \in \sigma \text{ and } f(x) \neq 0\} = f[\sigma] \setminus \{0\}$$

is a meet semilattice surjective homomorphism.

(ii) *If A is κ-partition complete, then B is κ-partition complete.*

Proof.

(i) The κ-completeness of f ensures that for any $\sigma \in \mathbb{P}_A^\kappa$ we have $\Sigma \hat{f}(\sigma) = f(\Sigma\sigma) = f(1) = 1$, whence $\hat{f}$ maps into $\mathbb{P}_B^\kappa$. The preservation of finite meets is inherited from f. To see that $\hat{f}$ is onto, let $\lambda < \kappa$ and suppose that $\tau = \{b_i : i \leq \lambda\}$ is a partition in B. We need to construct $\sigma \in \mathbb{P}_A^\kappa$ such that $\hat{f}(\sigma) = \tau$. For each $i \leq \lambda$ choose $a_i \in A$ such that $f(a_i) = b_i$, and define $c_i = a_i \cdot \Pi\{-a_j : j \leq \lambda,\ j \neq i\}$. Then $c_i \cdot c_j = 0$ for $i \neq j$, and since f is κ-complete, $f(c_i) = b_i \cdot \Pi\{-b_j : j \leq \lambda,\ j \neq i\} = b_i \neq 0$. Therefore $c_i \neq 0$ for all $i < \lambda$. Let $\sigma' = \{c_i : i \leq \lambda\}$ and $c = \Sigma\sigma'$. If $c = 1$ take $\sigma = \sigma'$, else take $\sigma = \sigma' \cup \{-c\}$. In either case, $\sigma \in \mathbb{P}_B^\kappa$ and $\hat{f}(\sigma) = \tau$.

(ii) Assume that A is κ-partition complete and let S be a subset of $\mathbb{P}^\kappa_B$ with $|S| < \kappa$. By part (i) we can choose for each $\tau \in S$ a partition $\sigma_\tau \in \mathbb{P}^\kappa_A$ such that $\hat{f}(\sigma_\tau) = \tau$. Let $\sigma = \bigotimes\{\sigma_\tau : \tau \in S\}$. It follows from the κ-completeness of f that $\hat{f}(\sigma)$ is the coarsest common refinement of S. □

Note that the converse of (ii) does not hold: take A to be any Boolean algebra that is not κ-partition complete and has at least one atom. Let B be the quotient of A modulo a principal (hence complete) ultrafilter. Then the canonical homomorphism from A to A/F is complete and $A/F \cong \mathbf{2}$ is κ-partition complete.

0.3 Ultrafilters and partitions

Our next proposition generalizes the well-known result about κ-complete ultrafilters in powerset algebras (see [1, p. 180]). Although it is probably known, we include a proof for the sake of completeness.

Proposition 0.9. *Let B be a κ-complete Boolean algebra and let F be an ultrafilter in B. Then the following are equivalent:*

(i) *F is κ-complete.*

(ii) *Any $\sigma \in \mathbb{P}^\kappa_B$ has exactly one element in common with F.*

Proof. Assume (i), and let $\sigma \in \mathbb{P}^\kappa_B$. Then $\Pi\{-x : x \in \sigma\} = 0$, whence it follows from the κ-completeness of F that $-x \notin F$ for some $x \in \sigma$. Since F is an ultrafilter, x is also an element of F, and since σ is an antichain, there is exactly one such x.

Conversely, assume (ii) and for some $\lambda < \kappa$ let $\{a_i : i < \lambda\}$ be a subset of F. If $a = \Pi_{i<\lambda} a_i \notin F$, then $-a = \Sigma_{i<\lambda}(-a_i) \in F$. Let $b_i = a_i \cdot (-\Sigma_{j<i} a_j)$. Then $\{b_i : i < \lambda\} \subset B$ is an antichain that joins to $-a$ and satisfies $b_i \leq -a_i$ for each $i < \lambda$. Consider the partition $\sigma = \{a\} \cup \{b_i : i < \lambda \text{ and } b_i \neq 0\}$. Since $a \notin F$, it follows from (ii) that $b_i \in F$ for some $i < \lambda$. This however implies $-a_i \in F$ which contradicts $a_i \in F$. □

Condition (ii) of the preceding lemma defines a function from $\mathbb{P}^\kappa_B$ to B^+, and the collection of all such functions is characterized by the following result.

Corollary 0.10. *Let B be a κ-complete Boolean algebra. Then B has a κ-complete ultrafilter iff there is a meet semilattice homomorphism $f : \mathbb{P}^\kappa_B \longrightarrow B^+$ such that $f(\sigma) \in \sigma$.* □

1 Partition completeness and weak representability

A Boolean algebra is said to be *κ-strongly representable* if it is isomorphic to a κ-set algebra, and *κ-weakly representable* if it is a κ-complete homomorphic image of a κ-set algebra. In this section we show that every κ-partition complete Boolean algebra is κ-weakly representable, and if κ is strongly inaccessible, then these two concepts coincide. We first observe that by Lemma 0.7(v)(vi), a Boolean algebra is κ-weakly representable if and only if it is κ-completely embedded into a κ-complete homomorphic image of a powerset algebra (or a κ-completely reduced power of $\mathbf{2}$).

Theorem 1.1. *Every κ-partition complete Boolean algebra is κ-weakly representable.*

Proof. Recall that for $\sigma \in \mathbb{P}^\kappa_B$, the powerset algebra $\mathcal{P}(\sigma)$ is (isomorphic to) a complete subalgebra of the κ-complete Boolean algebra B, and that if we identify $\mathcal{P}(\sigma)$ with this subalgebra, then B is an updirected union of the collection $\{\mathcal{P}(\sigma) : \sigma \in \mathbb{P}^\kappa_B\}$. An updirected union is of course a special case of a direct limit, and it was proved in [4] that every direct limit is embeddable into a reduced power. We wish to argue that, with a suitably chosen filter, this embedding is κ-complete. Let F be the filter in $\mathcal{P}(\mathbb{P}^\kappa_B)$ generated by the collection of principal ideals $\{(\sigma] : \sigma \in \mathbb{P}^\kappa_B\}$. Since the intersection of two principal ideals in $\mathbb{P}^\kappa_B$ is again principal, F is simply the collection of all subsets of $\mathbb{P}^\kappa_B$ that contain a principal ideal $(\sigma]$ for some $\sigma \in \mathbb{P}^\kappa_B$.

Suppose now that B is κ-partition complete. Then the generating collection for F, and hence F itself, is closed under meets of subsets with less than κ elements. Therefore F is κ-complete. Let C be the direct product of the algebras $\mathcal{P}(\sigma)$ for $\sigma \in \mathbb{P}^\kappa_B$, and define A to be the reduced product $\Pi_F\{\mathcal{P}(\sigma) : \sigma \in \mathbb{P}^\kappa_B\}$. Being a product of powerset algebras, C is of course isomorphic to a powerset algebra. By Lemma 0.7(iv), the canonical homomorphism $\pi : C \twoheadrightarrow A$ is κ-complete, hence A is κ-complete. The embedding f from B to A is defined as follows: For $b \in B$ consider the partition $\sigma = \{b, -b\}$ (or $\sigma = \{1\}$ if $b \in \{0, 1\}$), and for $\tau \in \mathbb{P}^\kappa_B$ let

$$\bar{b}(\tau) = \begin{cases} b & \text{if } \tau \leq \sigma, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\bar{b} \in C$, so we may define $f(b) = \bar{b}/F$. It remains to show that f is injective and preserves meets of subsets with less than κ elements (preservation of joins follows from this). Suppose $\bar{b}/F = 1/F$ for $b \in B$. Then the set $E = \{\tau : \bar{b}(\tau) = 1\}$ is in F, so there exists some partition $\varrho \in \mathbb{P}^\kappa_B$ such that $(\varrho] \subseteq E$. But for any common refinement τ of $\{b, -b\}$ and ϱ, we have $b = \bar{b}(\tau) = 1$, so the kernel of f is trivial. Now let S be a subset of B with $|S| < \kappa$, and define $b = \Pi\, S$ and $\varrho = \bigotimes\{\{a, -a\} \in \mathbb{P}^\kappa_B : a \in S\}$. The meets exist because B is κ-partition complete. We have to show that $f(b) = \Pi\, f[S]$. Since $a \in \mathcal{P}(\varrho)$ for all $a \in S$, it follows that $b \in \mathcal{P}(\varrho)$. This means that whenever $\tau \leq \varrho$, we get $\bar{b}(\tau) = b = \Pi\, S$, as well as $\bar{a}(\tau) = a$ for all $a \in S$. Hence $(\Pi_{a \in S}\bar{a})(\tau) = \Pi\, S = \bar{b}(\tau)$ and consequently $f(b) = \bar{b}/F = (\Pi_{a\in S}\bar{a})/F$. Since the canonical homomorphism π is κ-complete, the last result is equal to $\Pi_{a\in S}(\bar{a}/F) = f[S]$. □

The above theorem was originally proved by observing that B is isomorphic to the κ-bounded Boolean power $\mathbf{2}[B]^\kappa$ which is a direct limit and hence embeddable into a reduced power of $\mathbf{2}$ by [4]. A slight extension of this classical result shows that if B is κ-partition complete, then the embedding and the reduced power are κ-complete. For further details, the interested reader is refered to [11].

Theorem 1.2. *Let κ be a strongly inaccessible cardinal. Then for a Boolean algebra B, the following are equivalent:*

(i) *B is κ-partition complete.*

(ii) *B is κ-distributive.*

(iii) *B is κ-weakly representable.*

Proof. We first prove the equivalence of (i) and (ii), and then show that (iii) implies (i). The remaining implication follows from Theorem 1.1. Note that the equivalence of (ii) and (iii) was previously proved in [3] and [6].

By definition (i) implies (ii). To prove that (ii) implies (i), consider a subset S of $\mathbb{P}_B^\kappa$ with $|S| < \kappa$ and define $\lambda = |\bigcup S|$. By κ-distributivity, the meet σ of S exists in $\mathbb{P}_B$. We need to show that $\sigma \in \mathbb{P}_B^\kappa$. Since κ is strongly inaccessible we have $\lambda < \kappa$ and $|\sigma| \leq \lambda^{|S|} \leq (\lambda \cdot |S|)^{\lambda \cdot |S|} = 2^{\lambda \cdot |S|} < \kappa$. Therefore $\sigma \in \mathbb{P}_B^\kappa$ as required.

Suppose (iii) holds, let A be a κ-set algebra and let $f : A \longrightarrow B$ be a κ-complete surjective homomorphism. Since A is κ-distributive, it follows from the equivalence of (i) and (ii) that A is κ-partition complete. Now (i) follows by Proposition 08.(ii) since B is a κ-complete image of a κ-partition complete algebra. □

For any set X of size less than κ, it is of course obvious that $\mathcal{P}(X)$ is κ-partition complete. From the preceding theorem we get the following result for arbitrary powerset algebras.

Corollary 1.3. *Let κ be a strongly inaccessible cardinal. Then every κ-set algebra is κ-partition complete. In particular, if $\mathcal{P}(X)$ is any powerset algebra, then $\mathcal{P}(X)$ is κ-partition complete.* □

2 Atomless κ-partition complete Boolean algebras for regular κ

In this section we introduce basic κ-tree algebras for any infinite cardinal κ. In case κ is regular, they represent a simple way of constructing κ-partition complete algebras, and are in a sense the building blocks for all atomless κ-partition complete Boolean algebras (see Proposition 2.4 and Theorem 2.8).

Recall that a *tree* is a poset in which the set of predecessors of any element is wellordered. A *root* is an element with no predecessor, a *branch* is a maximal chain, and the *αth level* is the collection of elements for which the set of predecessors is order-isomorphic to the ordinal α. For more details on trees, the reader should consult [2], [5], [7] or [8].

A tree is said to be *normal* if

(N_1) it has a single root,

(N_2) each element has at least two immediate successors in the next level, and

(N_3) for every element x and every nonempty level σ, there is some element in σ that is comparable with x.

(A further normality condition given in [7] and [8] which says that 'distinct elements at a level indexed by a limit ordinal must have distinct sets of predecessors' is not required for our purposes.)

Let κ be an infinite cardinal. A *κ-tree* is a single rooted tree of height κ, in which each level has cardinality less than κ. If T is a normal κ-tree, then T is dually and densely embedded in a Boolean algebra B, the *Boolean completion of T* ((N_2) ensures that the dual order of the tree is separative). Under this embedding the root of the T becomes the top element of B and each level of T becomes a member of $\mathbb{P}_B^\kappa$ ((N_3) and T being dense in B imply that the levels of T are partitions). Instead of referring to dual trees and dual embeddings, it is convenient to assume henceforth that *trees are ordered in the opposite direction and that normal trees are dense subsets of their Boolean completion.*

Definition 2.1. Let T be a normal κ-tree, dually embedded in its Boolean completion B, and let L_T be the set of all levels in T. Since B is complete, for each $\sigma \in L_T$, the powerset algebra $\mathcal{P}(\sigma)$ is a complete subalgebra of B. Define

$$A_T = \bigcup\{\mathcal{P}(\sigma) : \sigma \in L_T\}.$$

Then A_T is a union of an increasing chain of powerset algebras, and since the height of T is an infinite cardinal, A_T is clearly atomless. We shall refer to A_T as the *basic κ-tree algebra of T*. More generally, a Boolean algebra A is called a *basic κ-tree algebra* if it is isomorphic to A_T for some normal κ-tree T. A *basic κ-tree subalgebra* of a Boolean algebra is any κ-complete subalgebra that is also a basic κ-tree algebra. (Note that the term "tree algebra" refers to a different construction in [7], hence the adjective "basic".)

Example 2.2. Recall that an atomless complete Boolean algebra B is a κ-Suslin algebra if it is κ-partition complete and $\mathbb{P}^\kappa_B = \mathbb{P}_B$ (see Example 0.4(iii)). If T is a normal κ-tree in which every antichain has less than κ elements, then the Boolean completion B of T is called a *κ-Suslin tree algebra*. Note that if κ is regular, then B is a basic κ-tree algebra: Since T is dense in B, for any $x \in B$ there exists a subset U of T such that $x = \Sigma U$ and the elements of U are pairwise disjoint (see [7, pp. 44, 54]). It follows from the regularity of κ that $U \subseteq \mathcal{P}(\sigma)$ for some $\sigma \in L_T$, whence $x \in A_T$.

For later reference, we make the following two simple observations.

Lemma 2.3.

(i) *Assume B is the Boolean completion of a normal κ-tree T. Then we have $B = \bigcup\{\mathcal{P}(\sigma) : \sigma \in \mathbb{P}^\kappa_B\}$, hence $B = A_T$ if and only if $\mathbb{P}^\kappa_B = \mathbb{P}^\kappa_{A_T}$.*

(ii) *If $\kappa = 2^{<\kappa}$ (in particular if κ is strongly inaccessible) then every basic κ-tree algebra has cardinality $\leq \kappa$.* □

Proposition 2.4. *For a regular cardinal κ let T be a normal κ-tree. Then*

(i) *L_T is a decreasing chain which is cofinal in $\mathbb{P}^\kappa_{A_T}$ and*

(ii) *A_T is a κ-partition complete Boolean algebra.*

Proof.

(i) Let σ be a partition of A_T, and assume that $|\sigma| < \kappa$. Since κ is a regular cardinal there exists a level $\tau \in L_T$ such that $\sigma \subseteq \mathcal{P}(\tau)$. Thus τ is a refinement of σ.

(ii) Let S be a subset of $\mathbb{P}^\kappa_{A_T}$ with $|S| < \kappa$. It follows from (i) that for every $\sigma \in S$ there is a $\tau_\sigma \in L_T$ such that τ_σ refines σ. Since κ is regular, there is a level $\varrho \in L_T$ such that $\{\tau_\sigma : \sigma \in S\} \subseteq \mathbb{P}_{\mathcal{P}(\varrho)}$, whence $S \subseteq \mathbb{P}_{\mathcal{P}(\varrho)}$. By Lemma 0.1(i) $\mathbb{P}_{\mathcal{P}(\varrho)}$ is a principal filter of $\mathbb{P}^\kappa_{A_T}$, so $\bigotimes S$ exists in $\mathbb{P}^\kappa_{A_T}$. □

Not every atomless κ-partition complete Boolean algebra is a basic κ-tree algebra. Let X be a set of cardinality at least κ, and consider the filter F of all subsets S such that $|X \setminus S| < \kappa$. Then F is $\mathrm{cf}(\kappa)$-complete and $|\mathcal{P}(X)/F| \geq 2^\kappa$ ([7, pp. 372–374]), hence if κ is strongly inaccessible, then by Lemma 2.3(ii), $\mathcal{P}(X)/F$ is not a basic κ-tree algebra. However it is atomless and κ-partition complete by Theorem 1.2.

It can be shown within ZFC (without assuming that κ is strongly inaccessible) that for any regular κ and $\gamma \geq \kappa$, there exists an atomless κ-partition complete Boolean algebra B with $|B| \geq \gamma$.

Recall that a subalgebra of a κ-complete Boolean algebra is *κ-completely generated by a set X* if it is the intersection of all κ-complete subalgebras that contain X. We note that, for regular cardinals κ, the basic κ-tree algebra A_T is κ-completely generated by T in the Boolean completion B of T, so it is a κ-complete subalgebra of B.

Lemma 2.5. *For a regular cardinal κ let B be an atomless κ-partition complete Boolean algebra and let $\mathcal{S}$ be a subset of $\mathbb{P}^{\kappa}_{B}$ with $|\mathcal{S}| \leq \kappa$. Then there exists a basic κ-tree subalgebra A of B such that $\mathcal{S} \subseteq \mathbb{P}^{\kappa}_{A}$.*

Proof. Let $\mathcal{S} = \{\sigma_i : i < \kappa\}$ be an enumeration of $\mathcal{S}$ (possibly with repetitions, to get a sequence of length κ). Since B is κ-partition complete, we can replace σ_i by $\bigotimes\{\sigma_j : j < i\}$, hence we may assume that the σ_i form a chain. Based on this chain, we define by induction a (dually) wellordered chain $C = \{\tau_i : i < \kappa\}$ such that $\bigcup C$ is a normal κ-tree with ith level τ_i.

Let $\tau_0 = \{1\}$. For a successor ordinal $i = j+1$, since B is atomless we can choose a partition $\tau_i' \in \mathbb{P}^{\kappa}_{B}$ such that $\tau_i' < \tau_j$ and $\tau_i' \cap \tau_j = \emptyset$. Now we let $\tau_i = \tau_i' \otimes \sigma_i$. For a limit ordinal i we define $\tau_i = \bigotimes\{\tau_j : j < i\} \otimes \sigma_i$. Under the induced order of B, the set $T = \bigcup C$ is clearly a κ-tree with a single root. Since $\sigma \cap \tau = \emptyset$ for distinct σ, $\tau \in C$, each element of C is a level of T and (N_2) holds. Finally (N_3) holds because C is ordered by refinement. Let A be the subalgebra that is κ-completely generated by T. Since A is κ-complete, $\mathcal{P}(\tau)$ is a complete subalgebra of A for every $\tau \in C$. It follows that $A = \bigcup\{\mathcal{P}(\tau) : \tau \in C\}$ because the union of all these powerset algebras contains T and is κ-complete (since κ is regular). Now the isomorphism between A and A_T is easily defined as the union of isomorphisms between corresponding subalgebras $\mathcal{P}(\tau)$ for each $\tau \in C$. Hence A is a basic κ-tree algebra. Since each partition in $\mathcal{S}$ is refined by some τ in C, we have $\mathcal{S} \subseteq \mathbb{P}^{\kappa}_{A}$. □

The next result characterises the basic κ-tree algebras internally. Since every κ-Souslin algebra is a basic κ-tree algebra, it is also interesting to view it in relation to [7, Theorem 14.20].

Theorem 2.6. *For a regular cardinal κ and a Boolean algebra B the following are equivalent:*

(i) *B is a basic κ-tree algebra.*

(ii) *B is an atomless κ-partition complete Boolean algebra that is κ-completely generated by a set of cardinality at most κ.*

(iii) *B is atomless, κ-partition complete and $\mathbb{P}^{\kappa}_{B}$ has a descending cofinal chain C of length κ (i. e., for each $\sigma \in \mathbb{P}^{\kappa}_{B}$ there exists $\tau \in C$ such that τ refines σ).*

Proof. (i) implies (iii) by Proposition 2.4, and (iii) implies (ii) since $\bigcup C$ has cardinality $\leq \kappa$ and κ-completely generates B. Now suppose (ii) holds and let S be the set which κ-completely generates B and has $|S| \leq \kappa$. Then $\{\{x, -x\} : x \in S\}$ is a collection of partitions in $\mathbb{P}^{\kappa}_{B}$ of cardinality at most κ. By the preceding lemma, there exists a basic κ-tree subalgebra A of B such that $\{x, -x\} \in \mathbb{P}^{\kappa}_{A}$ for all $x \in S$. So $S \subseteq A$, and since A is a κ-complete subalgebra, it follows that $A = B$. □

For example, since every atomless Boolean algebra is ω-partition complete, it follows from the above theorem that the countable atomless Boolean algebra is a basic ω-tree algebra (the only one up to isomorphism since atomless Boolean algebras are ω-categorical).

Corollary 2.7. *Let κ be a regular cardinal, and suppose B is a κ-partition complete Boolean algebra. Any family $\mathcal{F}$ of at most κ basic κ-tree subalgebras of B have a common extension which is also a basic κ-tree subalgebra.*

Proof. Since each $A \in \mathcal{F}$ is isomorphic to A_T for some normal κ-tree T, the image (under the isomorphism) of the levels of T form a chain C_A of partitions in $\mathbb{P}_B^\kappa$ of length κ. Let $\mathcal{S} = \bigcup\{C_A : A \in \mathcal{F}\}$. Then $|\mathcal{S}| = \kappa$ so by Lemma 2.5 there exists a basic κ-tree algebra that contains all partitions in $\mathcal{S}$. This algebra is a common extension of the algebras in $\mathcal{F}$, since it is a κ-complete subalgebra of B and each $A \in \mathcal{F}$ is κ-completely generated by $\bigcup C_A$. □

The following theorem is a direct consequence of Corollary 2.7.

Theorem 2.8. *For a regular cardinal κ let B be an atomless κ-partition complete Boolean algebra. Then B is a κ^+-updirected union of basic κ-tree algebras.* □

In Example 2.2 we observed that a κ-Suslin tree algebra is in fact a basic κ-tree algebra. Applying the preceding theorem to Suslin algebras, we obtain the following result.

Corollary 2.9. *A Boolean algebra is a κ-Suslin algebra if and only if it is a κ^+-updirected union of κ-Suslin tree algebras.* □

As noted above, the countable atomless Boolean algebra $F(\omega)$ is the only basic ω-tree algebra. Thus, for example, every atomless Boolean algebra is an updirected union of algebras that are isomorphic to $F(\omega)$. A. Pinus noted that this result also follows from the downward Löwenheim-Skolem theorem. Furthermore he pointed out the following application: The concept of a quasi-universal formula is defined in Mal'cev [9], and it is proved that quasi-universal formulas are preserved under updirected unions. Hence the preceding remark implies that any quasi-universal formula which holds in $F(\omega)$ will hold in all atomless Boolean algebras, in other words, the quasi-universal theory of atomless Boolean algebras is complete.

3 Almost compact cardinals

3.1 Cardinals with the tree property

A κ-tree is said to be *Aronszajn* if it does not have a branch of length κ. An infinite cardinal κ has the *tree property* if every κ-tree has a branch of length κ (i.e. is not an Aronszajn tree). The following proposition summarizes some known results about these cardinals.

Proposition 3.1

(i) *Every cardinal with the tree property is regular* ([8], p. 307).

(ii) *The König infinity lemma: ω has the tree property.* □

Lemma 3.2 ([8, p. 297). *Let κ be a regular infinite cardinal. If there exists an Aronszajn κ-tree, then there exists a normal Aronszajn κ-tree.* □

We are primarily interested in cardinals with the tree property. By Theorem 2.8, for regular κ, every atomless κ-partition complete Boolean algebra is an updirected union of basic κ-tree algebras. It turns out that κ has the tree property iff every basic κ-tree algebra has a κ-complete ultrafilter (see Proposition 3.4).

The proof of the following lemma is similar to the "ultrafilter proof" of the König infinity lemma.

Lemma 3.3. *Let κ be a regular cardinal and let T be a normal κ-tree. Then the basic κ-tree algebra A_T has a κ-complete ultrafilter iff T is a not an Aronzajn tree.*

Proof. In the forward direction this follows from Proposition 0.9 and Proposition 2.4(i). For the reverse direction, suppose T is not Aronszajn and let C be a branch of length κ in T. Consider any ultrafilter F in A_T which extends C. It follows from another application of Proposition 0.9 and Proposition 2.4(i) that F must be κ-complete. □

Combining the preceding lemma with Lemma 3.2 we obtain the following result.

Proposition 3.4. *Let κ be a regular cardinal. Then κ has the tree property iff every basic κ-tree algebra has a κ-complete ultrafilter.* □

We now recall some notions of compact cardinals (characterised by the properties we are interested in, rather than in terms of the original definitions). A cardinal κ is said to be

- *weakly compact* if it is strongly inaccessible and has the tree property,
- *measurable* if there exists a κ-complete nonprincipal ultrafilter in $\mathcal{P}(\kappa)$, and
- *strongly compact* if every proper κ-complete filter in any powerset algebra can be extended to a κ-complete ultrafilter.

It is well known that strong compactness implies measurability, which in turn implies weak compactness. There is also the notion of β-compactness which covers the whole range from measurability to strong compactness: For $\beta \geq \kappa$ we say that κ is *β-compact* if κ is strongly inaccessible and every κ-complete filter in $\mathcal{P}(\beta)$ that is generated by at most β sets can be extended to a κ-complete ultrafilter. It can be shown that κ is measurable iff it is κ-compact, and clearly κ is strongly compact iff it is β-compact for all $\beta \geq \kappa$.

Using the concept of κ-partition completeness, we now define another range of compactness notions which, in case κ is strongly inaccessible, cover the whole spectrum from weak to strong compactness.

Definition 3.5. For cardinals $\kappa \leq \gamma$ we say that κ is *γ-almost compact* if κ has the tree property and every κ-partition complete Boolean algebra with at most γ elements is κ-strongly representable.

The following characterisation theorem can be compared to similar characterizations of weakly compact cardinals (e. g. [2, p. 292]), with the noteworthy difference that we do not assume κ is strongly inaccessible.

Theorem 3.6. *Let κ be a cardinal with the tree property. Then for any $\gamma \geq \kappa$ the following conditions are equivalent:*

(i) *κ is γ-almost compact.*

(ii) *In any κ-partition complete Boolean algebra of cardinality at most γ, every proper principal filter can be extended to a κ-complete ultrafilter.*

(iii) *Every atomless κ-partition complete Boolean algebra of cardinality at most γ has a κ-complete ultrafilter.*

(iv) *Every κ-partition complete Boolean algebra of cardinality at most γ has a κ-complete ultrafilter.*

(v) *If B is a κ-partition complete Boolean algebra of cardinality at most γ, then there is a meet semilattice homomorphism $f : \mathbb{P}^{\kappa}_{B} \longrightarrow B^{+}$ such that for all $\sigma \in \mathbb{P}^{\kappa}_{B}$, $f(\sigma) \in \sigma$.*

(vi) *Let A be a κ-set algebra which is κ-generated by a set of at most γ elements. Let F be a κ-complete filter in A such that A/F is κ-partition complete and $|A/F| \leq \gamma$. Then F can be extended to a κ-complete ultrafilter.*

Proof. The equivalence of (i) and (ii) is a consequence of the well-known result that a Boolean algebra is κ-strongly representable iff it is κ-complete and every proper principal filter can be extended to a κ-complete ultrafilter (see e.g. [7, Theorem 0.5]). Clearly (iii) and (iv) are equivalent since if a Boolean algebra has an atom, then the principal filter generated by that atom is complete. The equivalence of (iv) and (v) follows from Corollary 0.10. The implication (ii) $\Rightarrow$ (iv) holds trivially, so we complete the proof by establishing that (iv) implies (vi) and (vi) implies (ii).

By (iv) the quotient algebra has a κ-complete filter D. Let $f : A \longrightarrow A/F$ be the canonical homomorphism. By Lemma 0.7(i) f is κ-complete. Hence $f^{-1}[D]$ is a κ-complete ultrafilter which extends F and therefore (vi) holds.

Finally, assume (vi). Since B is κ-distributive, by [7, Theorem 0.8], there is a κ-set algebra A, κ-generated by a set of $|B|$ elements, and a κ-complete surjective homomorphism $g : A \longrightarrow B$. For $a \in B^{+}$ let H be the principal filter generated by a, and let $f : B \longrightarrow B/H$ be the canonical homomorphism. It follows from Lemma 0.7(i) that f is κ-complete. Thus by Proposition 0.8(ii), B/H is κ-partition complete. Now $h = f \circ g$ is a κ-complete homomorphism from A onto B/H. Letting $F = \ker h$, we invoke (v) to obtain a κ-complete ultrafilter U which contains F. It follows that $h[U]$ is a κ-complete ultrafilter in B/H and therefore $f^{-1}[h[U]]$ is a κ-complete ultrafilter in B which contains a. Thus (ii) holds and the proof of the theorem is complete. □

Corollary 3.7. *Let κ be a cardinal with the tree property and assume that $\gamma = \kappa^{<\gamma}$. Then the following conditions are equivalent:*

(i) *κ is γ-almost compact.*

(ii) *If A is a κ-partition complete κ-set algebra such that $|A| \leq \gamma$, then every κ-complete proper filter in A can be extended to a κ-complete ultrafilter.*

Proof. Suppose κ is γ-almost compact, and let F be a κ-complete filter in a κ-partition complete κ-set algebra A with $|A| \leq \gamma$. By Proposition 0.8(ii), A/F is also κ-partition complete. Since $|A/F| \leq \gamma$, it follows from Theorem 3.6(iv) that A/F has a κ-complete ultrafilter U. Since the canonical homomorphism $f : A \longrightarrow A/F$ is κ-complete, $f^{-1}[U]$ is a κ-complete ultrafilter which extends F. Therefore (ii) is satisfied.

Conversely, we will show that (ii) implies Theorem 3.6(vi). So let A be a κ-set algebra generated by at most γ elements. Since $\gamma = \kappa^{<\gamma}$ we have $|A| \leq \gamma$. Thus for any κ-complete filter F in A we have $|A/F| \leq \gamma$. It follows from (ii) that any such filter can be extended to a κ-complete ultrafilter, whence Theorem 3.6(vi) holds. □

3.2 Between weak and strong compactness

The following theorem shows that restricted to strongly inaccessible cardinals, γ-almost compactness covers the whole range from weakly compact to measurable to strongly compact cardinals.

Theorem 3.8. *Let κ be a strongly inaccessible cardinal. Then we have:*

(i) *κ is weakly compact iff κ is κ-almost compact.*

(ii) *For $\beta \geq \kappa$, if κ is 2^β-almost compact, then κ is β-compact (in particular, if κ is 2^κ-almost compact, then κ is measurable).*

(iii) *κ is strongly compact iff κ is γ-almost compact for all $\gamma \geq \kappa$.*

Proof.

(i) By Corollary 2.7, any atomless κ-partition complete Boolean algebra with at most κ elements is a basic κ-tree algebra, so it follows from Proposition 3.4 and Theorem 3.6(iii) that κ is κ-almost compact iff κ has the tree property. Since κ is assumed to be strongly inaccessible, κ has the tree property iff κ is weakly compact.

(ii) Let $\beta \geq \kappa$ and suppose κ is 2^β-almost compact. Assuming that F is a proper κ-complete filter in $\mathcal{P}(\beta)$ that is generated by at most β sets, we need to show that F can be extended to a κ-complete ultrafilter. By Lemma 0.7(i), the canonical homomorphism $\pi : \mathcal{P}(\beta) \longrightarrow \mathcal{P}(\beta)/F$ is κ-complete. Since κ is strongly inaccessible, by Corollary 1.3, $\mathcal{P}(\beta)$ is κ-partition complete, hence $\mathcal{P}(\beta)/F$ is also κ-partition complete by Lemma 0.8(ii). Since $|\mathcal{P}(\beta)/F| \leq 2^\beta$, it follows from Theorem 3.6(v) that there exists a κ-complete ultrafilter U in $\mathcal{P}(\beta)/F$. Now $\pi^{-1}[U]$ is the desired κ-complete ultrafilter in $\mathcal{P}(\beta)$ which extends F.

(iii) follows from (ii). □

4 Some open problems

Problem 4.1.

(i) Consider the statement "There is a (regular) infinite cardinal κ and a cardinal $\gamma \geq \kappa$ such that κ is γ-almost compact". For which κ and γ is this statement provable (refutable or independent) in ZFC?

(ii) For which $\gamma \geq \kappa$ is γ-almost compactness indescribable?

Problem 4.2. If κ is strongly inaccessible then by Theorem 1.2, the concepts κ-partition completeness, κ-distributivity and κ-weak representability coincide. Let κ be an infinite regular cardinal such that these concepts coincide. Is κ strongly inaccessible? If not, is the existence of such an "almost inaccessible" cardinal provable (refutable or independent) in ZFC?

Problem 4.3. By Theorem 3.8(ii), if κ is strongly inaccessible and 2^κ-almost compact, then κ is measurable. Is the converse false?

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